Support Vector Machine

Han Sol Kang

Contents

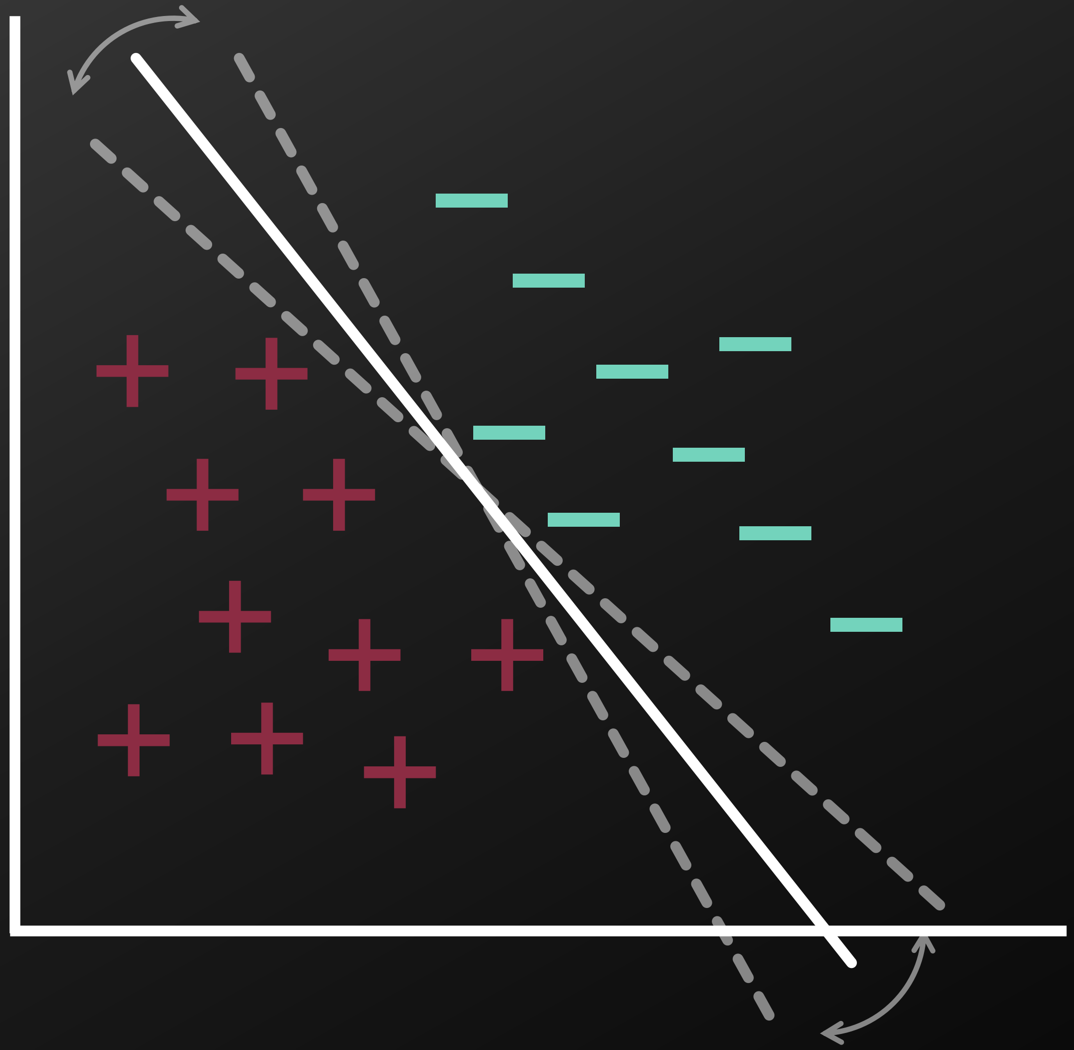
Introduction

[Vladimir N. Vapnik]

Vladimir Naumovich Vapnik is one of the main developers of the Vapnik–Chervonenkis theory of statistical learning, and the **co-inventor of the support vector machine method**.

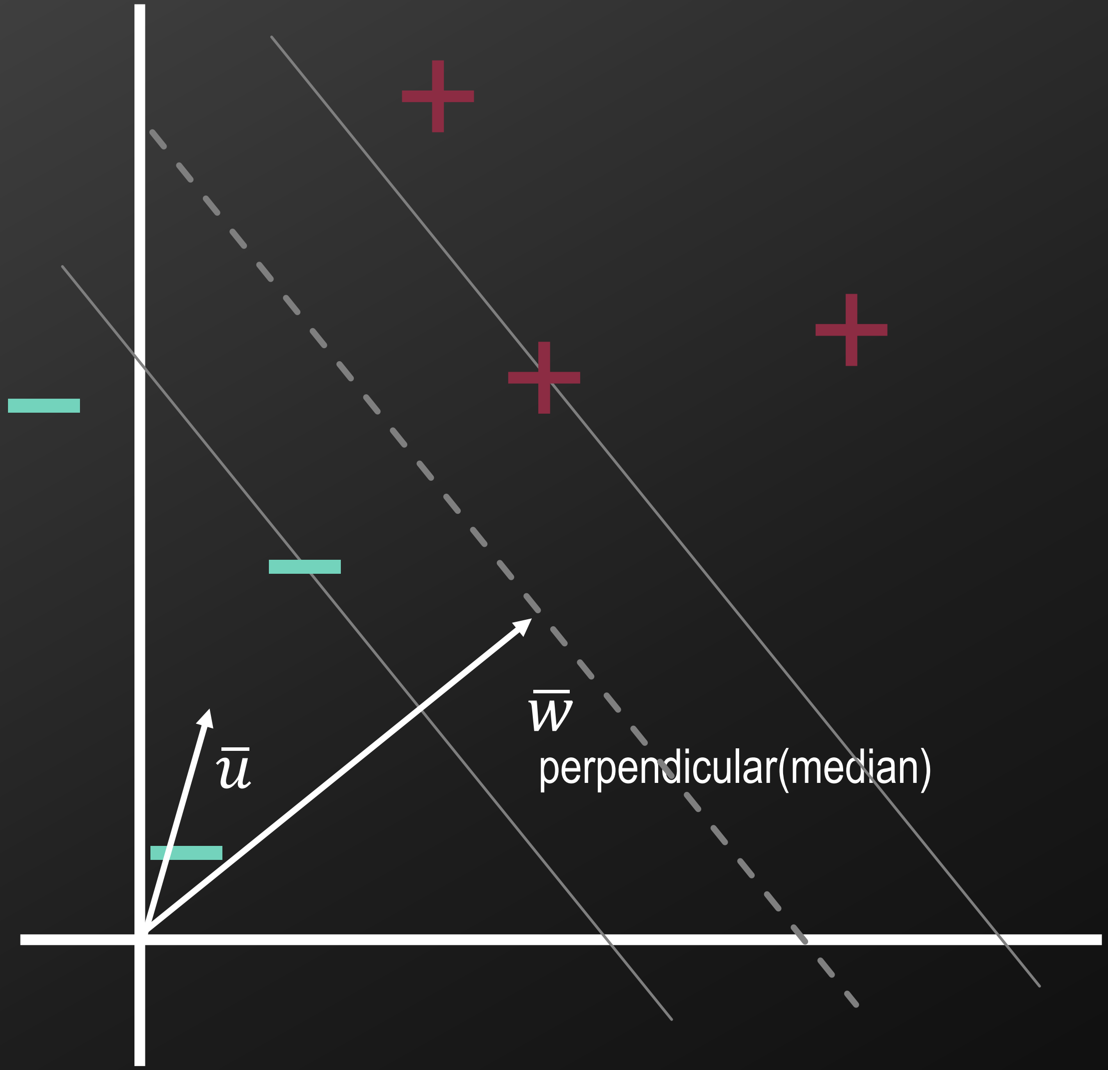
Vladimir Vapnik was born in the Soviet Union.(Dec 6, 1936)
He received Ph.D in statistics at the Institute of Control Sciences, Moscow in 1964.

Introduction



SVM (Hard margin)

❖ **Decision rule & Constraint.**



$\bar{w} \cdot \bar{u} \geq C$

$\bar{w} \cdot \bar{u} + b \geq 0$ then + **Decision Rule**

$\bar{w} \cdot \bar{x}_{+} + b \geq 1$

$\bar{w} \cdot \bar{x}_{-} + b \leq -1$

$y_i = \begin{cases} +1 & \text{for + samples} \\ -1 & \text{for - samples} \end{cases}$

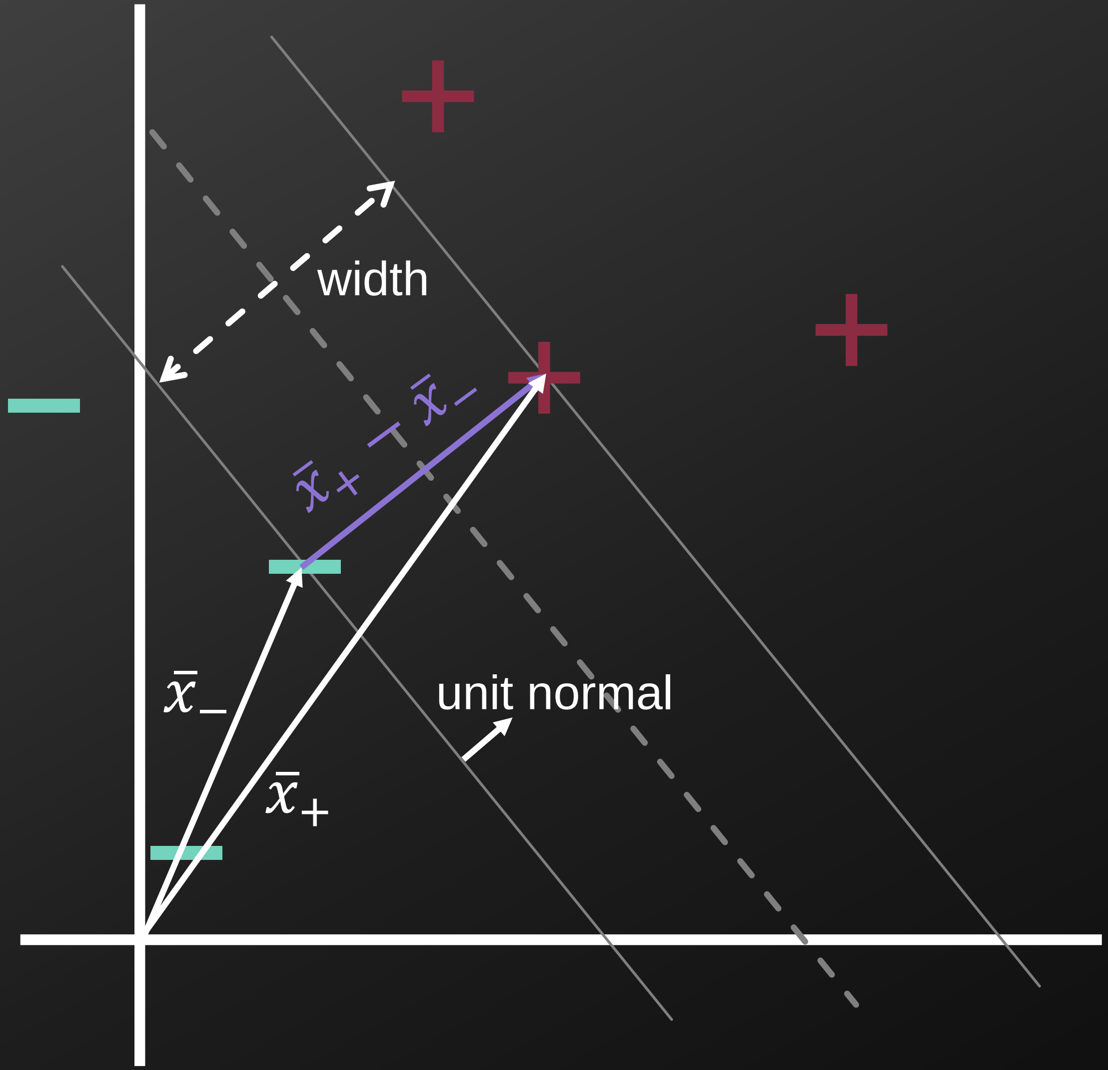
$y_i(\bar{w} \cdot \bar{x}_i + b) \geq 1$

$y_i(\bar{w} \cdot \bar{x}_i + b) - 1 \geq 0$ **Constraint**

$y_i(\bar{w} \cdot \bar{x}_i + b) - 1 = 0$ **Gutter**

SVM (Hard margin)

❖ Width & Margin



$$\text{width} = (\bar{x}_+ - \bar{x}_-) \cdot (unit\ normal)$$

$$= (\bar{x}_+ - \bar{x}_-) \cdot \frac{\bar{w}}{\|\bar{w}\|}$$

$$= \bar{x}_+ \cdot \frac{\bar{w}}{\|\bar{w}\|} - \bar{x}_- \cdot \frac{\bar{w}}{\|\bar{w}\|}$$

Use the gutter condition.

$$y_i(\bar{w} \cdot \bar{x}_i + b) - 1 = 0$$

+ samples : $\bar{w} \cdot \bar{x}_+ + b - 1 = 0$

\- samples : $-\bar{w} \cdot \bar{x}_- - b - 1 = 0$

$$= \frac{-b+1}{\|\bar{w}\|} + \frac{b+1}{\|\bar{w}\|} = \frac{2}{\|\bar{w}\|}$$

SVM (Hard margin)

❖ Maximize(Minimize) Problem.

Width : $\frac{2}{\|\bar{w}\|}$

Maximize

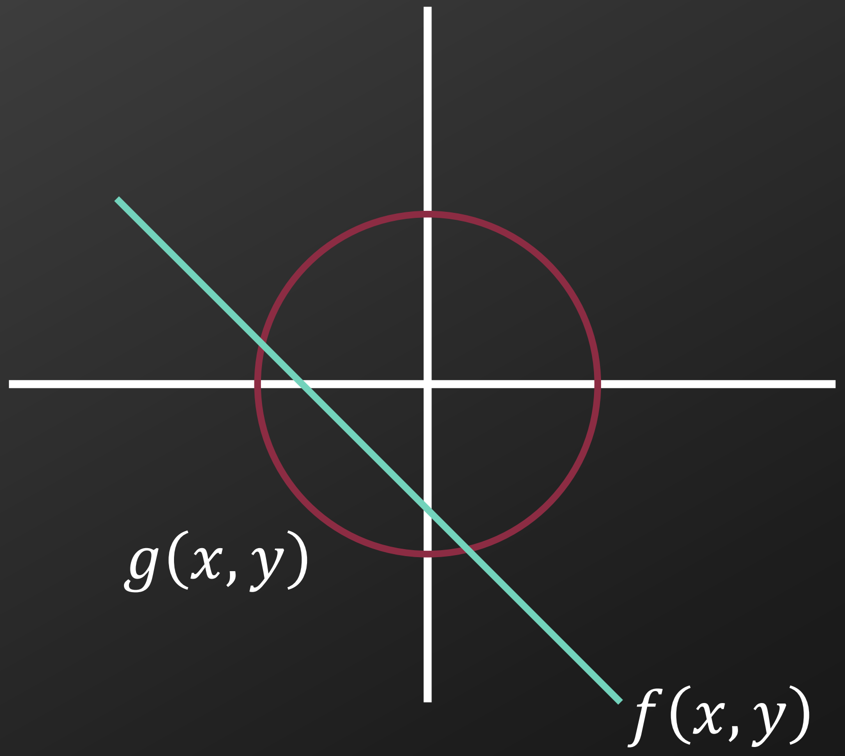
Margin : $\frac{1}{\|\bar{w}\|}$ ➡ $\|\bar{w}\|$ | $\frac{1}{2}\|\bar{w}\|^2$ + $y_i(\bar{w} \cdot \bar{x}_i + b) - 1 \geq 0$

Maximize | **Minimize** | **Minimize** | **Constraint**

Solver : Lagrange multiplier method

SVM (Hard margin)

❖ **Lagrange multiplier method.**



$$\nabla f(x,y) = \alpha \nabla (g(x,y))$$

$$\nabla f(x,y) - \alpha \nabla (g(x,y)) = 0$$

$$\mathrm{L}(x,y,\alpha) = f(x,y) - \alpha(g(x,y))$$

$$\mathrm{L}(x,y,\alpha) = f(x,y) - \sum_{i=1}^{n} \alpha_i (g_i(x,y))$$

$$\nabla \mathrm{L}(x,y,\alpha) = 0$$

SVM (Hard margin)

❖ KKT(Karush–Kuhn–Tucker) Condition

$$\mathrm{L}(\theta, \alpha) = f(\theta) - \sum_{i=1}^{n} \alpha_i (g_i(\theta)) \qquad g_i(\theta) \geq 0$$

KKT Condition 1

$$\frac{\partial L}{\partial \theta_i} = 0, \qquad i = 1, \ldots, k$$

KKT Condition 2

$$\alpha_i \geq 0, \qquad i = 1, \ldots, k$$

KKT Condition 3

$$\alpha_i g_i\,(\theta) = 0, \qquad i = 1, \ldots, k$$

SVM (Hard margin)

❖ Finding Optimal Hyperplane

$$L = \frac{1}{2}\|\bar{w}\|^2 - \sum_{i=1}^{N} \alpha_i \ [y_i(\bar{w} \cdot \bar{x}_i + b) - 1]$$

$$= \frac{1}{2}\|\bar{w}\|^2 - \sum \alpha_i[y_i(\bar{w} \cdot \bar{x}_i + b) - 1]$$

KKT Condition 1

$$\frac{\partial L}{\partial \bar{w}} = \bar{w} - \sum \alpha_i y_i \bar{x}_i = 0$$

$$\bar{w} = \sum \alpha_i y_i \bar{x}_i$$

$$\frac{\partial L}{\partial b} = -\sum \alpha_i y_i = 0$$

$$\sum \alpha_i y_i = 0$$

KKT Condition 2

$$\alpha_i \geq 0$$

KKT Condition 3

$$\alpha_i(y_i(\bar{w} \cdot \bar{x}_i + b - 1) = 0$$

SVM (Hard margin)

❖ **Finding Optimal Hyperplane**

$$L = \frac{1}{2}\left(\sum \alpha_i y_i \bar{x}_i\right)\left(\sum \alpha_j y_j \bar{x}_j\right) - \sum \alpha_i y_i \bar{x}_i \left(\sum \alpha_j y_j \bar{x}_j\right) - b\left(\sum \alpha_i y_i\right) + \sum \alpha_i$$

$$= \sum \alpha_i - \frac{1}{2}\left(\sum \alpha_i y_i \bar{x}_i\right)\left(\sum \alpha_j y_j \bar{x}_j\right)$$

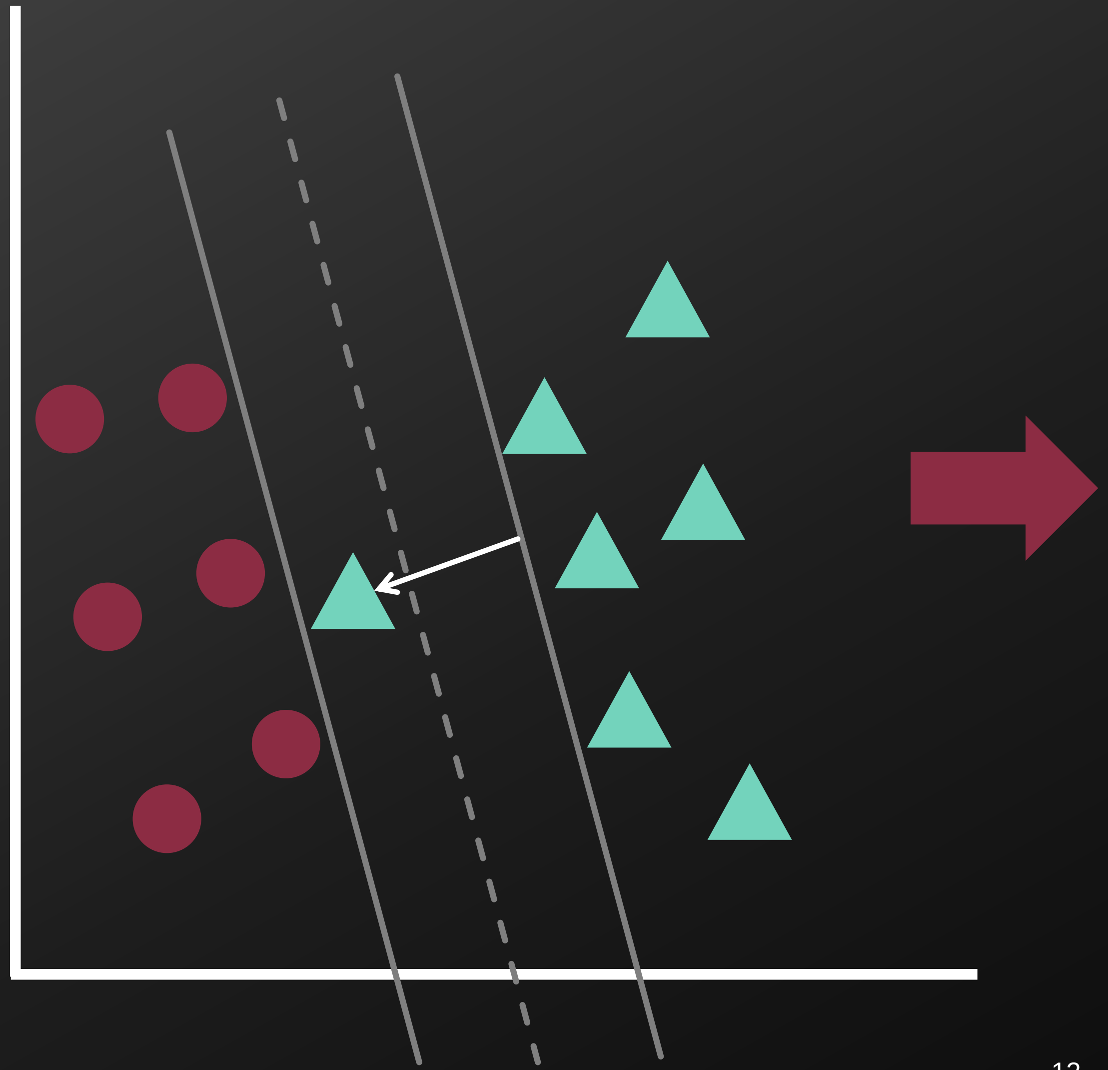
$$= \sum \alpha_i - \frac{1}{2}\sum_i \sum_j \alpha_i \alpha_j y_i y_j (\bar{x}_i \cdot \bar{x}_j) \qquad \sum \alpha_i y_i = 0 \quad \alpha_i \geq 0$$

Quadratic programming

$$\overline{w^*} = \sum \alpha_i{}^* y_i \bar{x}_i \qquad \overline{w^*} \cdot \bar{x}_{s+} + b^* = +1$$

SVM (Hard margin)

❖ Problem of hard margin method

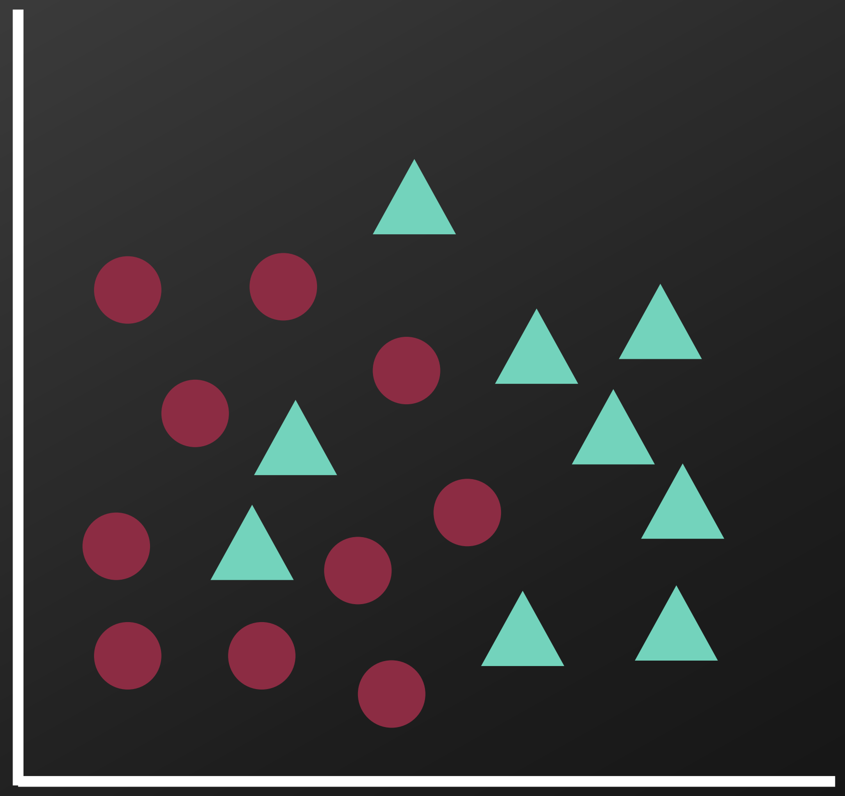


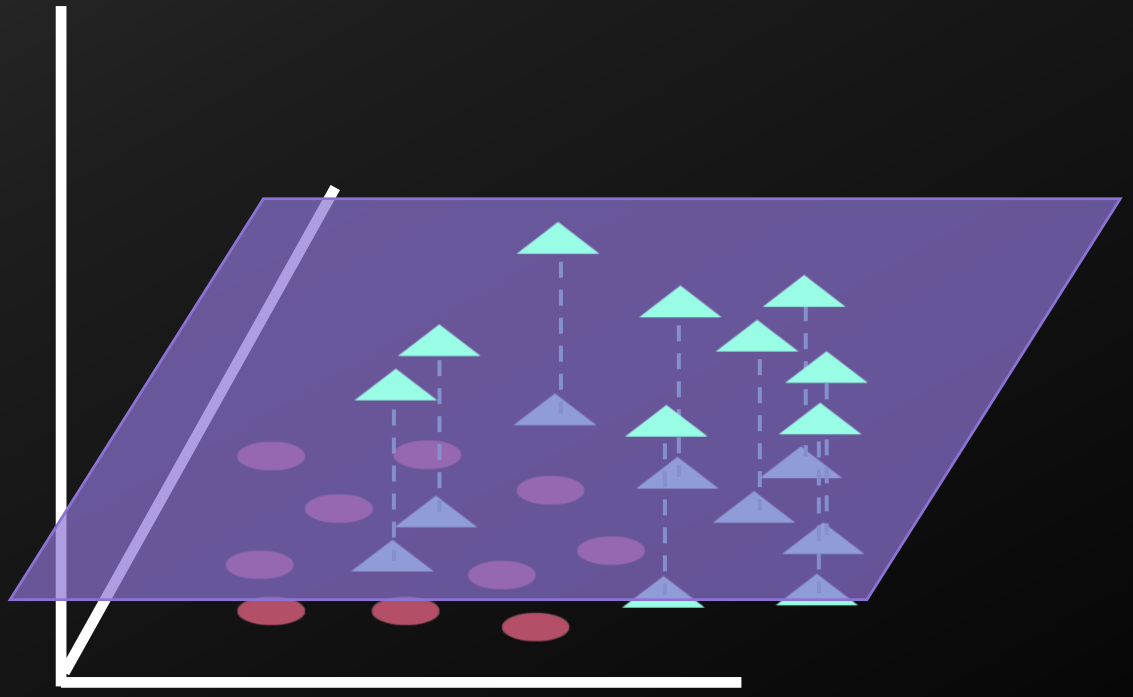
1) Soft Margin with slack variable

2) Kernel trick

Kernel Trick

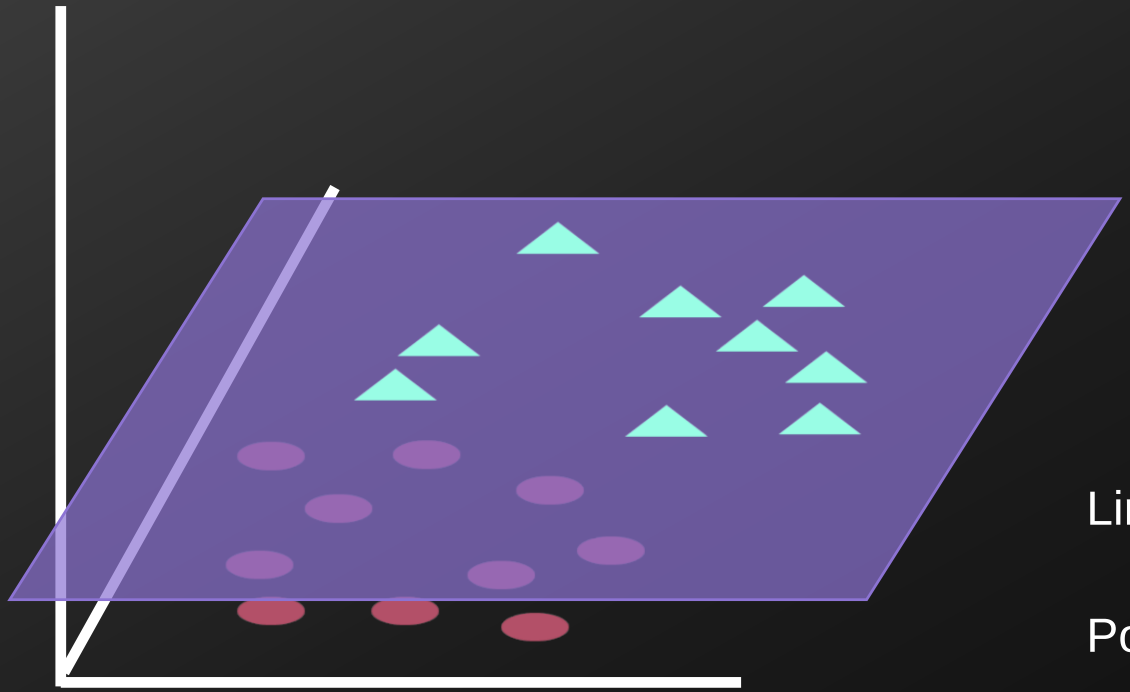
❖ **Concept of Kernel trick**





Kernel Trick

❖ Concept of Kernel trick



$$L = \sum \alpha_i - \frac{1}{2} \sum_i \sum_j \alpha_i \alpha_j y_i y_j (\bar{x}_i \cdot \bar{x}_j)$$

$$\Phi(\bar{x}_i) \cdot \Phi(\bar{x}_j)$$

$$k(\bar{x}_i, \bar{x}_j) = (\Phi(\bar{x}_i) \cdot \Phi(\bar{x}_j))$$

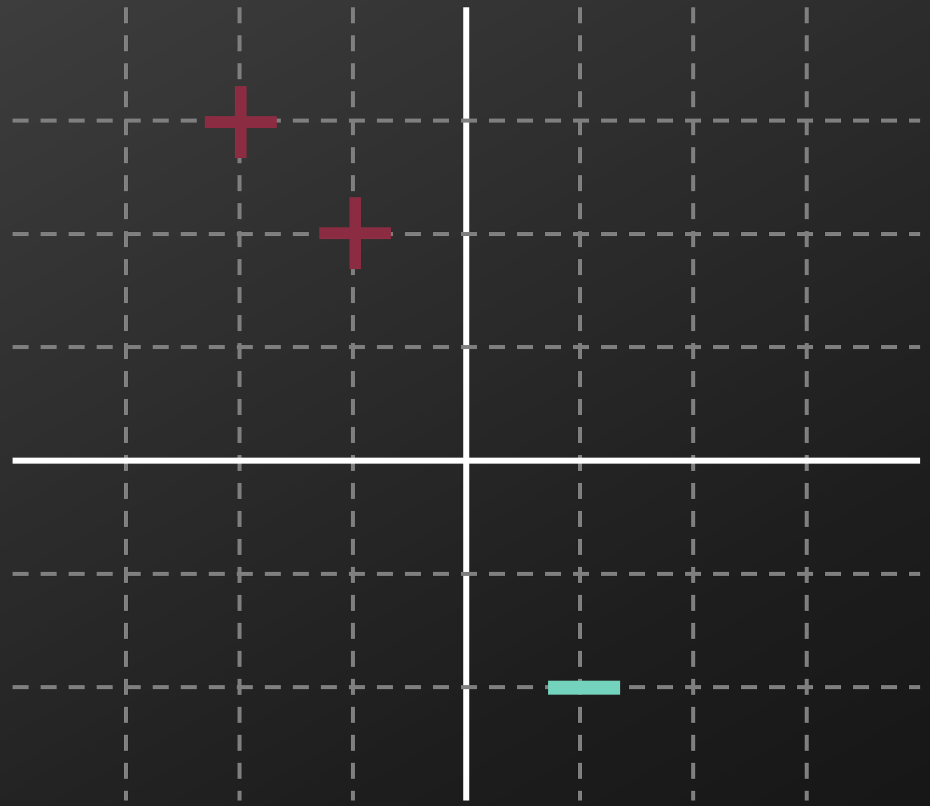
Linear : $k(\bar{x}_i, \bar{x}_j) = \bar{x}_i \cdot \bar{x}_j$

Polynomial : $k(\bar{x}_i, \bar{x}_j) = (1 + \bar{x}_i \cdot \bar{x}_j)^n$

Gaussian : $k(\bar{x}_i, \bar{x}_j) = e^{\frac{-\|\bar{x}_i - \bar{x}_j\|^2}{2\sigma^2}}$

Example

❖ SVM(Hard margin)



$$x_1 = (-1,2),\ \ x_2 = (-3,3)$$
$$x_3 = (1,-2)$$

$$\mathrm{L} = \sum \alpha_i - \frac{1}{2}\sum_i \sum_j \alpha_i \alpha_j y_i y_j (\bar{x}_i \cdot \bar{x}_j)$$

$y_i y_j (\bar{x}_i \cdot \bar{x}_j)$

5	9	5
9	18	9
5	9	5

$$= \alpha_1 + \alpha_2 + \alpha_3 - \frac{5}{2}\alpha_1{}^2 - 9\alpha_2{}^2 - \frac{5}{2}\alpha_3{}^2$$
$$- 9\alpha_1\alpha_2 - 5\alpha_1\alpha_3 - 9\alpha_2\alpha_3$$

Example

- SVM(Hard margin)

$$L = \alpha_1 + \alpha_2 + \alpha_3 - \frac{5}{2}{\alpha_1}^2 - 9{\alpha_2}^2 - \frac{5}{2}{\alpha_3}^2 - 9\alpha_1\alpha_2 - 5\alpha_1\alpha_3 - 9\alpha_2\alpha_3$$

$$\sum \alpha_i y_i = \alpha_1 + \alpha_2 - \alpha_3 = 0 \quad \alpha_1 \geq 0, \alpha_2 \geq 0, \alpha_3 \geq 0$$

from KKT condition 3

$$\alpha_i(y_i(\overline{w} \cdot \bar{x}_i + b - 1) = 0$$

case 1) $\alpha_1 = 0$, case 2) $\alpha_2 = 0$, case 3) $\alpha_3 = 0$, case 4) $\alpha_1 \neq 0, \alpha_2 \neq 0, \alpha_3 \neq 0$

Example

❖ SVM(Hard margin)

Case 1) $\alpha_1 = 0$

$$\alpha_2 - \alpha_3 = 0, \quad \alpha_2 = \alpha_3 \qquad L = \alpha_1 + \alpha_2 + \alpha_3 - \frac{5}{2}{\alpha_1}^2 - 9{\alpha_2}^2 - \frac{5}{2}{\alpha_3}^2 - 9\alpha_1\alpha_2 - 5\alpha_1\alpha_3 - 9\alpha_2\alpha_3$$

$$\mathrm{L} = 2\alpha_2 - \frac{41}{2}{\alpha_2}^2$$

$$\frac{\partial L}{\partial \alpha_2} = 2 - 41\alpha_2 = 0 \qquad \boxed{\alpha_2 = 0.049, \alpha_3 = 0.049, \alpha_1 = 0}$$

$$\bar{w} = \sum \alpha_i y_i \bar{x}_i = 0.049[(-3,3) - (1,-2)\,] = 0.049\,(-4,5) = \boxed{(-0.196,0.245)}$$

$$\alpha_i[y_i(\bar{w} \cdot \bar{x}_i + b) - 1] = 0.049[(-0.196,0.245) \cdot (-3,3) + b - 1] \; = 0$$

$$b = 1 - (-0.196,0.245) \cdot (-3,3) = 1 - 1.323 = \boxed{-0.323}$$

Example

- SVM(Hard margin)

$\overline{w} = (-0.196, 0.245) \quad b = -0.323$

$\overline{w} \cdot x_1 + b \geq 1 \quad (-0.196, 0.245) \cdot (-1, 2) - 0.323 = 0.363$ **Not satisfied**

$\overline{w} \cdot x_2 + b \geq 1 \quad (-0.196, 0.245) \cdot (-3, 3) - 0.323 = 1$ **SV**

$\overline{w} \cdot x_3 + b \leq -1 \quad (-0.196, 0.245) \cdot (1, -2) - 0.323 = -1$ **SV**

Example

❖ SVM(Hard margin)

Case 2) $\alpha_2 = 0$

$\alpha_1 - \alpha_3 = 0, \quad \alpha_1 = \alpha_3$

$$L = \alpha_1 + \alpha_2 + \alpha_3 - \frac{5}{2}{\alpha_1}^2 - 9{\alpha_2}^2 - \frac{5}{2}{\alpha_3}^2 - 9\alpha_1\alpha_2 - 5\alpha_1\alpha_3 - 9\alpha_2\alpha_3$$

$$\mathrm{L} = 2\alpha_1 - 10{\alpha_1}^2$$

$$\frac{\partial L}{\partial \alpha_1} = 2 - 20\alpha_1 = 0$$

$$\boxed{\alpha_1 = 0.1, \alpha_3 = 0.1, \alpha_2 = 0}$$

$$\bar{w} = \sum \alpha_i y_i \bar{x}_i = 0.1[(-1,2) - (1,-2)\,] = 0.1\,(-2,4) = \boxed{(-0.2,0.4)}$$

$$\alpha_i[y_i(\bar{w} \cdot \bar{x}_i + b) - 1] = 0.1[(-0.2,0.4) \cdot (-1,2) + b - 1] = 0$$

$$b = 1 - (-0.2,0.4) \cdot (-1,2) = 1 - 1 = \boxed{0}$$

Example

❖ SVM(Hard margin)

$\overline{w} = (-0.2,0.4) \quad b = 0$

$\overline{w} \cdot x_1 + b \geq 1 \quad (-0.2,0.4) \cdot (-1,2) = 1$ **SV**

$\overline{w} \cdot x_2 + b \geq 1 \quad (-0.2,0.4) \cdot (-3,3) = 1.8$ **Satisfied**

$\overline{w} \cdot x_3 + b \leq -1 \quad (-0.2,0.4) \cdot (1,-2) = -1$ **SV**

Case 3) $\alpha_3 = 0$

$\alpha_1 + \alpha_2 = 0,$ **Impossible case**

Example

❖ SVM(Hard margin)

Case 4) $\alpha_1 \neq 0, \alpha_2 \neq 0, \alpha_3 \neq 0$

$$L = \alpha_1 + \alpha_2 + \alpha_3 - \frac{5}{2}{\alpha_1}^2 - 9{\alpha_2}^2 - \frac{5}{2}{\alpha_3}^2 - 9\alpha_1\alpha_2 - 5\alpha_1\alpha_3 - 9\alpha_2\alpha_3$$

$$\frac{\partial L}{\partial \alpha_1} = 1 - 5\alpha_1 - 9\alpha_2 - 5\alpha_3 = 0$$

$$\frac{\partial L}{\partial \alpha_2} = 1 - 18\alpha_2 - 9\alpha_1 - 9\alpha_3 = 0$$

$$\frac{\partial L}{\partial \alpha_3} = 1 - 5\alpha_3 - 5\alpha_1 - 9\alpha_2 = 0$$

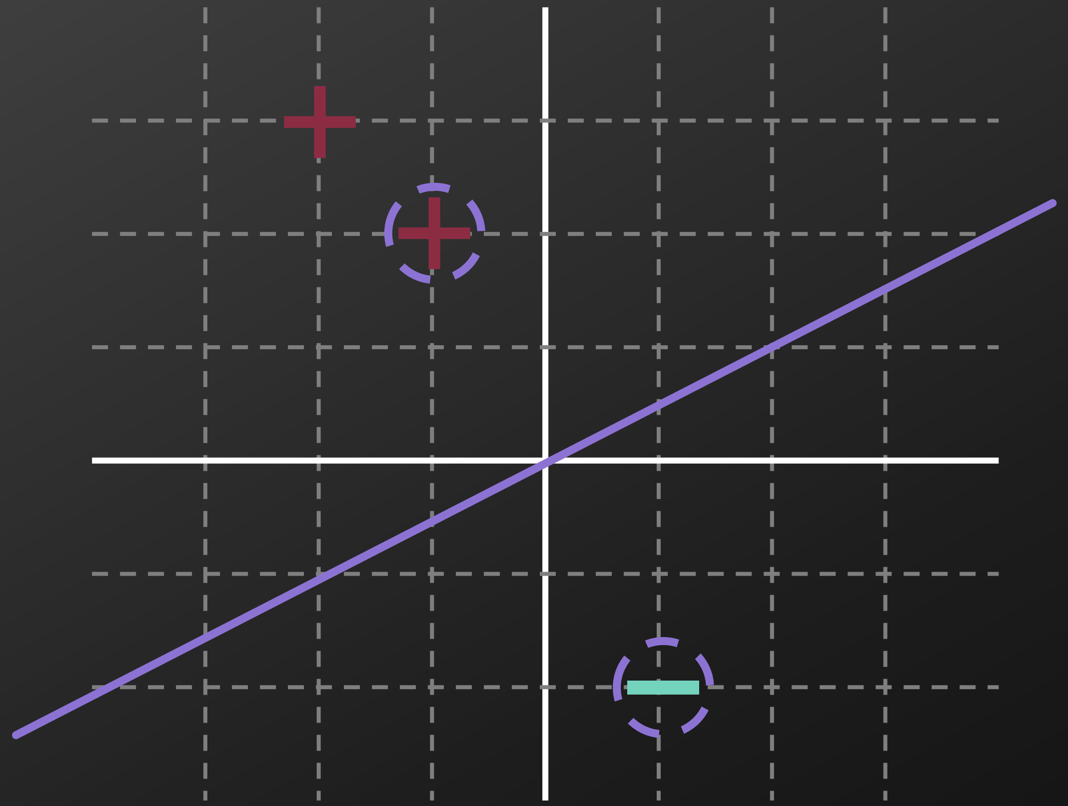
$$\begin{bmatrix} 5 & 9 & 5 \\ 9 & 18 & 9 \\ 5 & 9 & 5 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} 5 & 9 & 5 \\ 9 & 18 & 9 \\ 5 & 9 & 5 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.5 \\ -0.444 \\ 0.5 \end{bmatrix}$$

Impossible case

Example

❖ SVM(Hard margin)



$\bar{w} = (-0.2, 0.4) \quad b = 0$

$\bar{w} \cdot (x, y) + b = 0$

$-0.2x + 0.4y = 0 \quad y = 0.5x$

Example

❖ SVM(Hard margin)

```
X=[-1 2;-3 3;1 -2];
Y=['p';'p';'n'];

SVMModel=fitcsvm(X,Y);

sv = SVMModel.SupportVectors;
figure
gscatter(X(:,1),X(:,2),Y)
hold on
plot(sv(:,1),sv(:,2),'ko','MarkerSize',10)
legend('+','-','Support Vector')
grid on;
xlim([-5 5])
ylim([-5 5])
line([-5 5],[0 0])
line([0 0],[-5 5])

w=SVMModel.Beta
b=SVMModel.Bias

x=linspace(-5,5,100);
y=-1*(w(1)/w(2))*x+b;
plot(x,y)
```

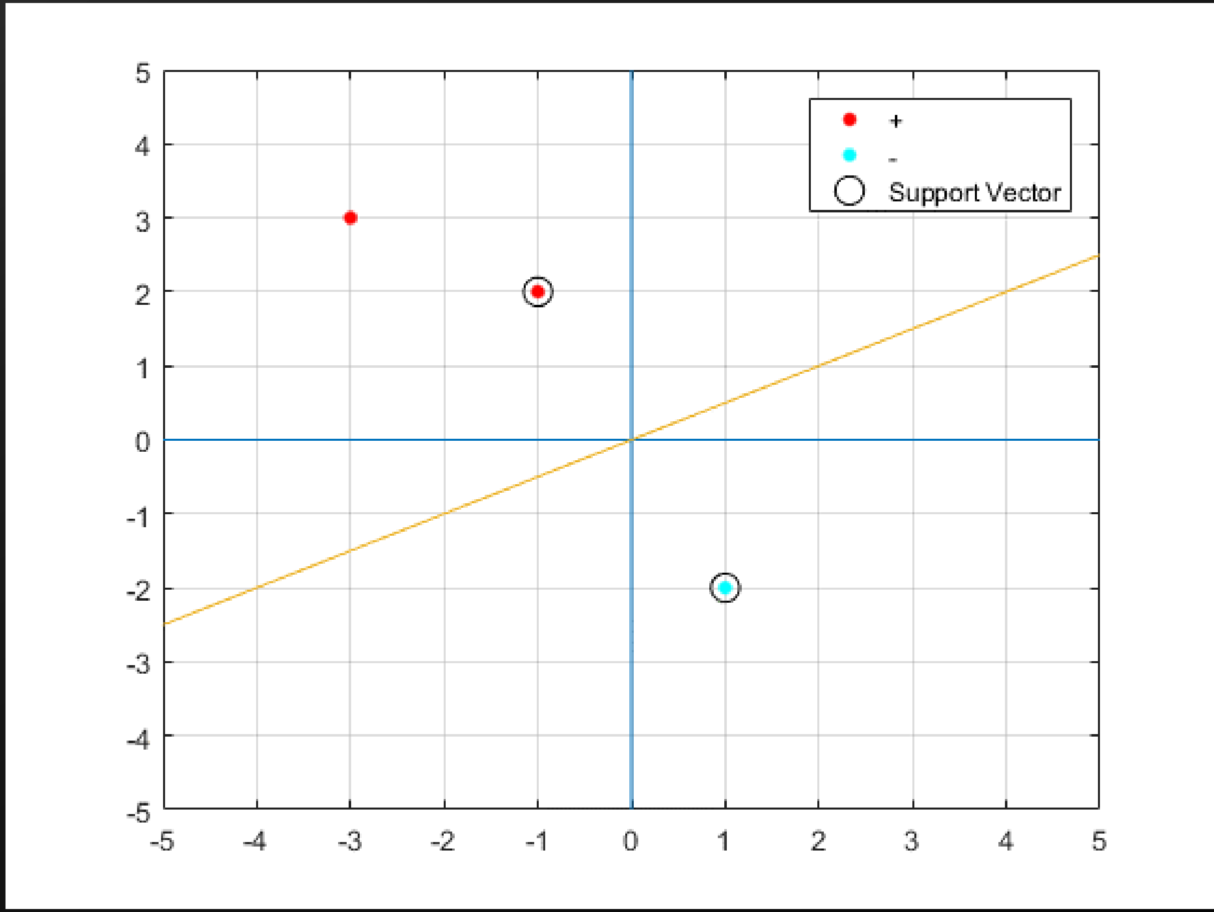
```
>> svm_test1

w =

  -0.2000
   0.4000


b =

     0
```



Q & A

Thank You!!!